Maximum and Minimum Stress

Week 9:
Transformation of stresses and strains

1. Principal and maximum stresses
2. Principal stresses in 3D

Transformation of stress and strain

Principal and maximum stresses

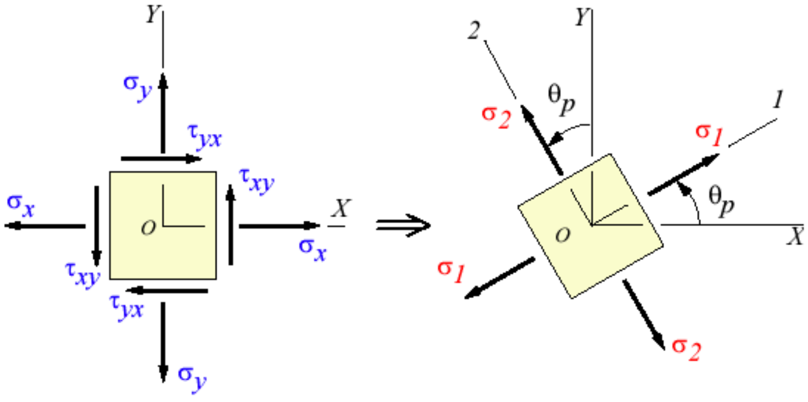
- From the equations for normal and shear stress under an arbitrary angle, we can see that there are angles of maximum and minimum shear and normal stresses
- We can calculate these angles by setting the respective derivatives to zero
- For the maximum/minimum of the normal stresses we get:

$$\left(\sigma_{x'}\right)_{\substack{\max \\ \min}} = \sigma_{1 \text{ or } 2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- This is the *principal stress* and the angle under which it is is the *principal axis*

EPFL

Transformation to principal stresses



Stresses in given coordinate system

Principal stresses

- Assume an element is under a combination of normal and shear stresses when looked at in a specific coordinate system.
- There exists a rotated coordinate system in which the description of the same stress element will result in only normal stresses, with the shear stresses being zero.
- The normal stresses expressed in this rotated coordinate system are the *principal stresses*. One normal stress is the maximum normal stress. The other normal stress is the minimal stress
- The axes of this rotated coordinate system are the *principal axes*.

Principal and maximum shear stresses

- For the plane where the shear stress is maximum we get:

$$\tau_{\substack{\max \\ \min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

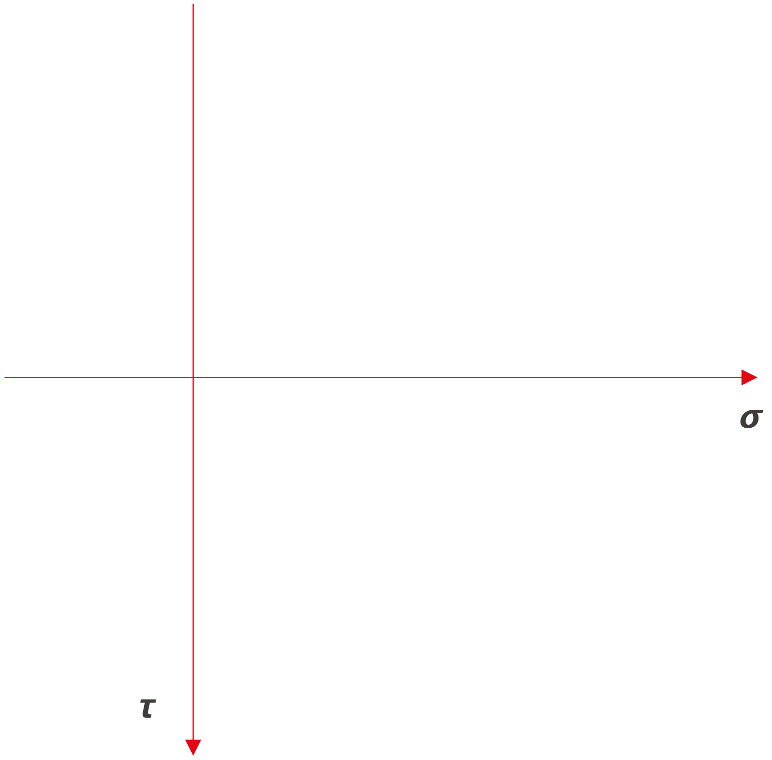
- The absolute value of the maximum shear stress is the same for the axis of maximum and the axis of minimum shear stress. This is understandable, since the material doesn’t care if it is “sheared left or right”
- In the principal axis, there is nor shear stress
- In the axis of maximum shear stress, there is also a normal stress (average normal stress)

$$\sigma_{\theta_S} = \frac{\sigma_x + \sigma_y}{2}$$

	Normal Stress	Shear stress
Angle Maximum	$\theta_N = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}\right)$	$\theta_S = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_{xx}-\sigma_{yy}}{2\tau_{xy}}\right)$
Max/Min Value	$\sigma_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$	$\tau_{max,min} = \pm\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$
"Other" stress at that angle	$\tau_{xy} = 0$	$\sigma_{xy'} = \sigma_{av} = \frac{\sigma_{xx}+\sigma_{yy}}{2}$

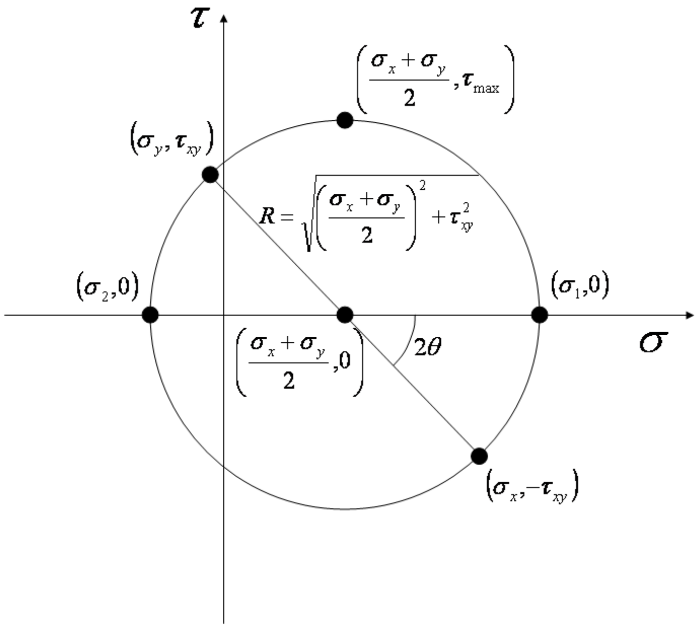
Mohr's circle of stress

- The symmetry in the equations for the shear stress and the normal stress suggest that there is an easy way to describe their relationship, and to calculate the normal and shear stresses in any direction
- We plot for each direction we've calculated the σ_i and τ_i on a coordinate system of σ and τ
- We know form our calculations:
  - There are directions where σ is maximum or minimum and τ=0
  - There are also directions where τ is maximum or minimum and $\sigma=(\sigma_x+\sigma_y)/2$
- What do we get if we draw all possible combinations on here?



Georg Fantner

What we can learn from Mohr's circle of stress



σ_1 is the maximum normal stress, σ_2 is the minimum normal stress, and there are no shear stresses in that direction

The largest shear stress is equal to the radius of the circle and in the direction of max shear stress we have a normal stress of $\sigma_{av}=(\sigma_1+\sigma_2)/2$

If $\sigma_x+\sigma_y=0$, then there is an axis of pure shear stress

The sum of all stresses in any two mutually orthogonal directions planes is constant

Transformation of plane strain

- Since both stress and strain are tensors, we can treat the coordinate transform of the strain in a similar way as that for the stress

$$\overleftrightarrow{\varepsilon}' = \mathbf{Q} \cdot \overleftrightarrow{\varepsilon} \cdot \mathbf{Q}^{\mathbf{T}}$$

- The transformation equations for plane strain then are:

$$\boxed{\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta) + \frac{\gamma_{xy}}{2}\sin(2\theta)}$$

$$\boxed{\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta) - \frac{\gamma_{xy}}{2}\sin(2\theta)}$$

$$\boxed{\gamma_{x'y'} = -(\varepsilon_x - \varepsilon_y)\sin(2\theta) + \gamma_{xy}\cos(2\theta)}$$

Principal strains

- These formulas are very similar to the ones we've derived for stress
- We can therefore again show that we can plot all the possible combinations of normal strain and shear strain in a graph with axes ε,***γ/2***, and obtain a circle: *Mohr's circle of strain*
- By setting the derivatives of transformation expressions for normal strain and shear strain with respect to θ to zero, we can again calculate the principal strains:

$$\boxed{(\varepsilon_{x'})^{max}_{min} = \varepsilon_1 \& \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}}$$

- At the principal angle:

$$\boxed{\tan(2\theta) = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}}$$

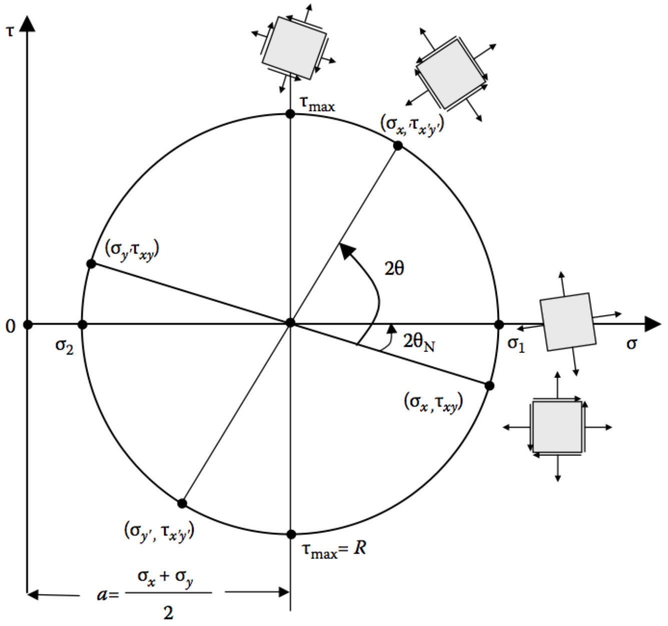
Mohr's circle of strain in 2D

$$(\varepsilon_{av}, 0) = \left(\frac{\varepsilon_x - \varepsilon_y}{2}, 0\right)$$

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\left(\frac{\gamma_{xy}}{2}\right)_{max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$(\varepsilon_{x'})^{max}_{min} = \varepsilon_1 \& \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$



Transformation of plane stress & strain in 2D

Summary

	Plane stress	Plane strain
max normal	$(\sigma_{x'})_{min}^{max} = \sigma_{1\&2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$	$(\varepsilon_{x'})_{min}^{max} = \varepsilon_{1\&2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$
max shear	$(\tau_{xy})_{min}^{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$	$\left(\frac{\gamma_{xy}}{2}\right)_{max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$
Angle max normal	$\theta_N = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\right)$	$\theta_N = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}\right)$
Angle max shear	$\theta_S = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}\right)$	$\theta_N = \frac{1}{2}\tan^{-1}\left(-\frac{\varepsilon_{xx} - \varepsilon_{yy}}{\gamma_{xy}}\right)$

Principle stresses in 3D

- The stress tensor is a symmetric 3x3 tensor that can be written in different coordinate systems.
- From linear algebra we know that one coordinate system exists in which the tensor only has non-zero elements in its diagonal (everywhere else the components are zero).

$$\overleftrightarrow{\tau} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \overset{\text{in principal coord.}}{\Longrightarrow} \begin{pmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{y'x'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{z'y'} & \sigma_{z'} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

- The axes of this coordinate system are the *principal axes*
- The elements in the diagonal are the *principal stresses*
- When the stress tensor is represented in its principal coordinate system, there are no shear stresses, only normal stresses

Principle stresses in 3D

Calculating the principal stresses

- Calculating the principal stresses equal finding the eigenvalues and eigenvectors of the stress tensor:

$$det\left(\overleftrightarrow{\sigma} - \lambda\overleftrightarrow{E}\right) = 0$$

- When we know the 3D stress state in our reference coordinate system, we can calculate the principal stresses by calculating the roots of the characteristic equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

- With I_1, I_2, I_3:

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 \end{aligned}$$

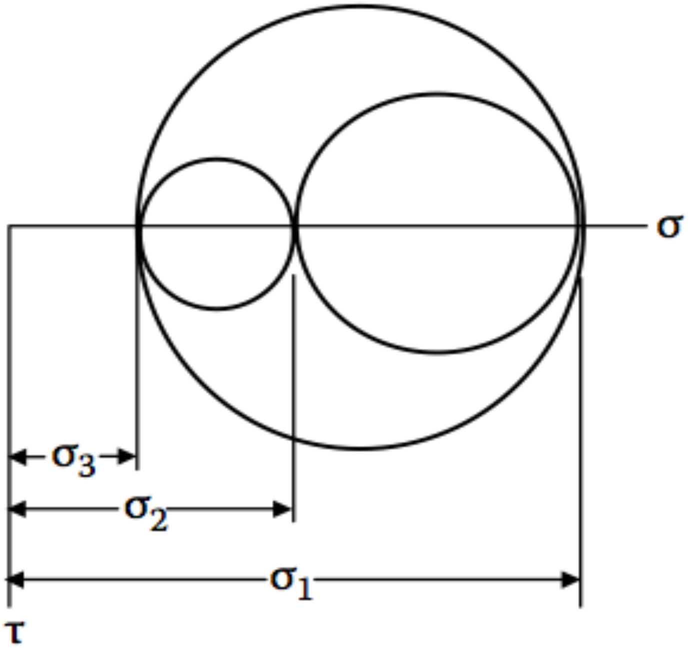
- I_1, I_2, I_3 are the stress invariants.

Principle stresses in 3D

- The stress invariants in the principal axes are then:

$$
\begin{aligned}
I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \\
I_3 &= \sigma_1\sigma_2\sigma_3
\end{aligned}
$$

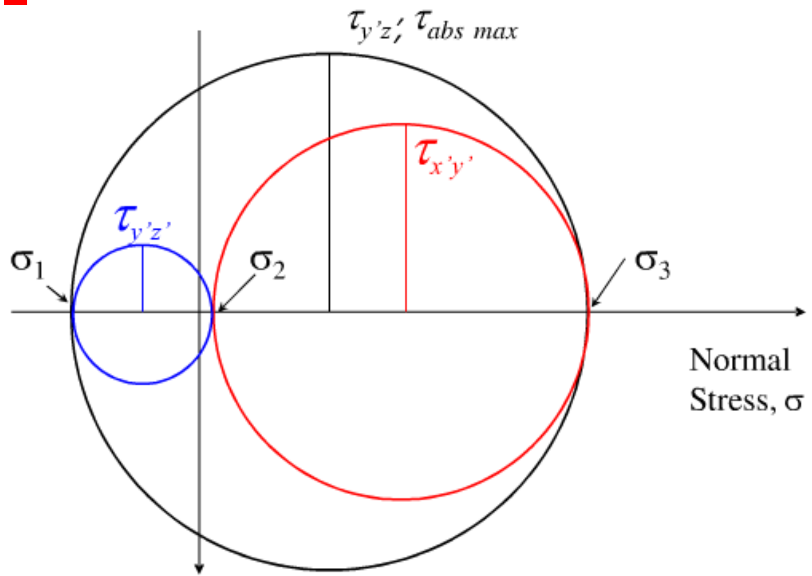
- With the *eigenvalues* of the 3D stress tensor we can then calculate the Eigenvectors. The Eigenvectors point in the direction of the principal axes of the stress state.



Mohr’s circle in 3D

- The stress tensor is dependent only on the stress state, and not on our initial choice of coordinate system.
- We’ve previously learned to draw the Mohr’s circle in 2D. Those were in essence projection of the 3D stress state in 2D
- To get to Mohr’s circle in 3D, we can therefore draw three individual Mohr’s circles for the planes x-y, x-z, and y-z, as long as we know the principal stresses

Mohr's circle in 3D- Maximum shear stress



$$\tau_{max,3} = \pm\frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{max,2} = \pm\frac{\sigma_1 - \sigma_3}{2}$$

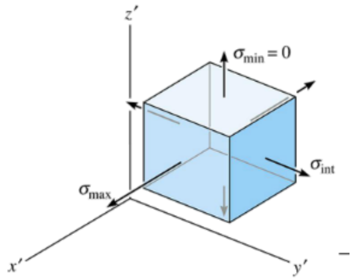
$$\tau_{max,1} = \pm\frac{\sigma_2 - \sigma_3}{2}$$

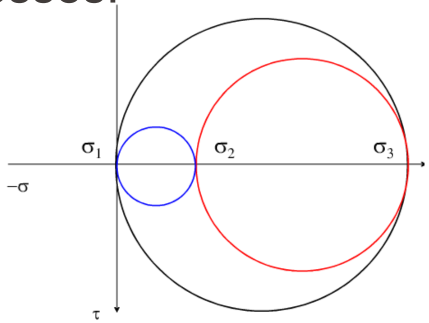
Comment: Sometimes we use the opposite numbering convention $\sigma_3<\sigma_2<\sigma_1$

We can use Mohr's circle in 3D to evaluate what the maximum shear stresses are in the 3 principal directions

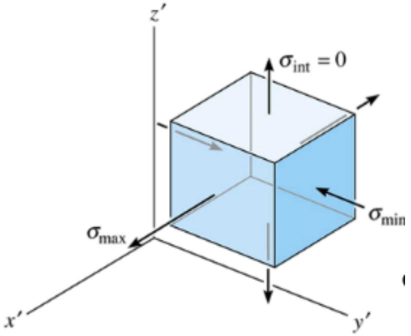
Mohr's circle in 3D - 3D state of plane stress

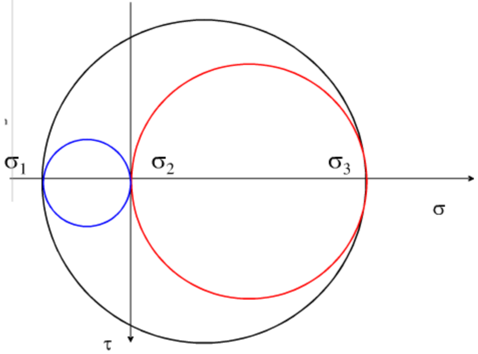
- 3D state of plane stress – 2 positive stresses:





- 3D state of plane stress – 1 positive stress, 1 negative:





Example: Triaxial stress state – NOT plane stress

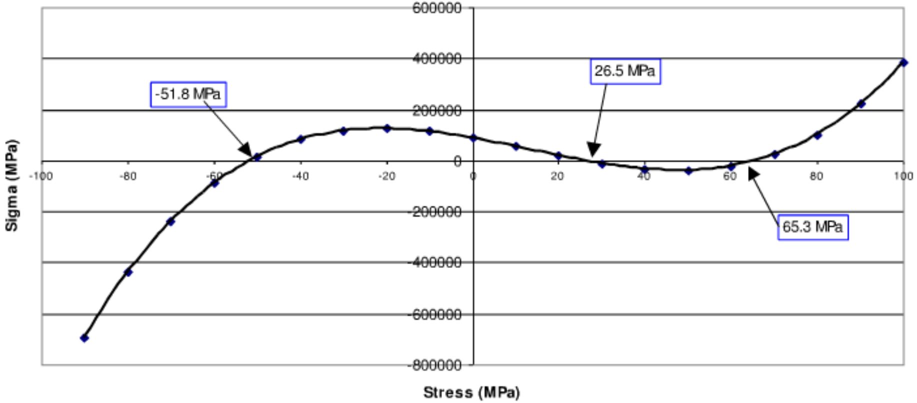
$$\overleftrightarrow{\tau} = \begin{pmatrix} 20 & 40 & -30 \\ 40 & 30 & 25 \\ -30 & 25 & -10 \end{pmatrix} MPa$$

Calculate the maximum principal stresses and maximum shear stresses for the stress state on the left.

Solution:

- Calculate stress invariants
- Calculate roots of characteristic equation (through a plot)
- Extract the maximum shear and principal stresses

Example: Triaxial stress state – NOT plane stress



$$\sigma_3 = 65.3 MPa$$

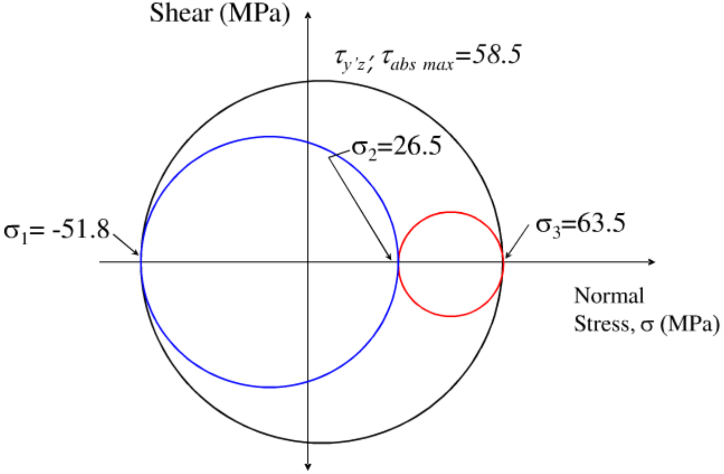
$$\sigma_2 = 26.5 MPa$$

$$\sigma_1 = -51.8 MPa$$

$$\tau_{\max} = 1/2(65.3 + 51.8)$$

$$= 58.5 MPa$$

Example: Triaxial stress state – NOT plane stress



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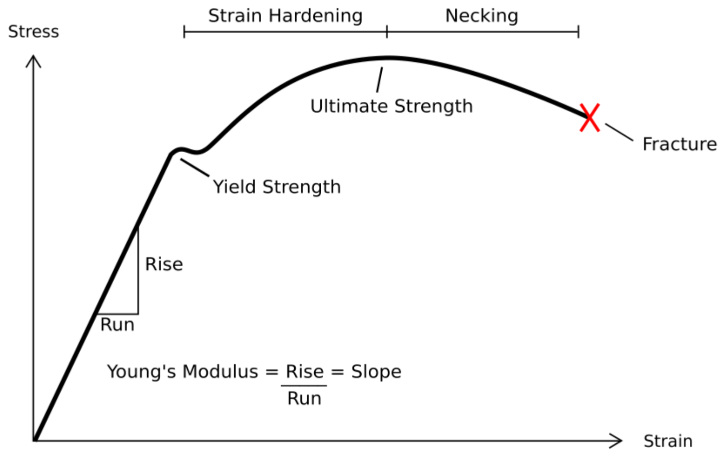
Failure criteria and beam bending



What is Failure?

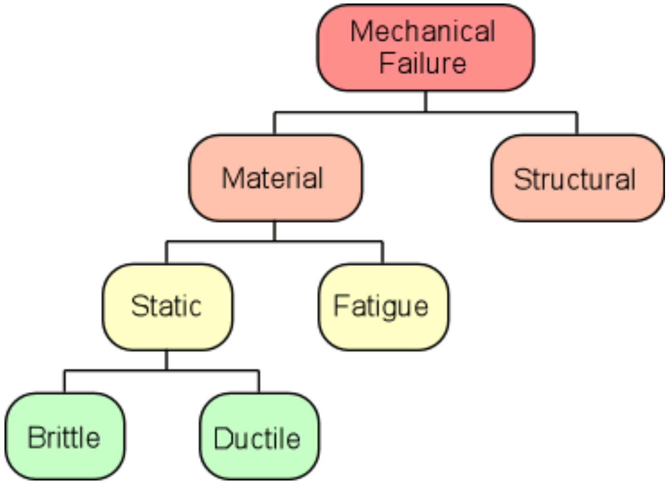
Failure – any change in a machine part which makes it unable to perform its intended function.(From Spotts M. F. and Shoup T. E.)

We will normally use a **yield failure criteria** for **ductile materials**. The ductile failure theories presented are based on yield.

Failure Theories

- Static failure
  - Ductile
  - Brittle
  - Stress concentration

- Recall
  - Ductile
    - Significant plastic deformation between yield and fracture
  - Brittle
    - Yield ~= fracture



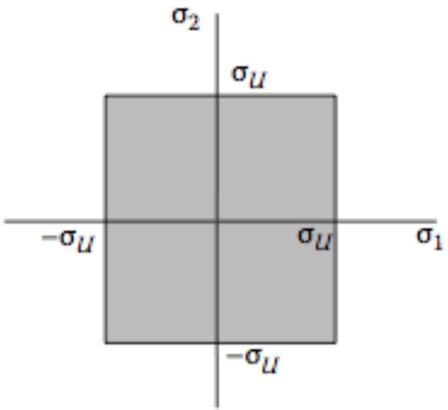
Failure of brittle materials

- A brittle material subjected to uniaxial tension fails without necking, on a plane normal to the material’s long axis
- Under uniaxial tensile stress, the normal stress that causes it to fail is the ultimate tensile strength of the material
- If the material is under three-dimensional stress state, it is useful to determine the principal stresses at any given point and to use one of the failure criteria

Failure of brittle materials

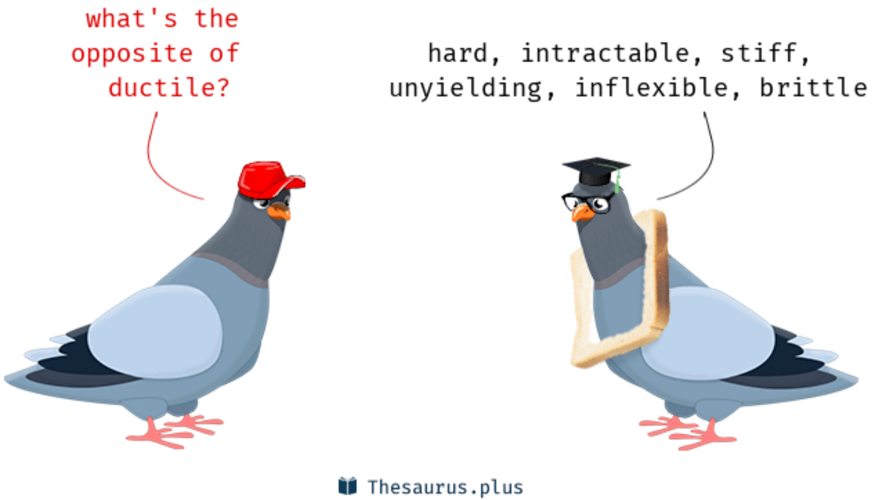
Maximum normal stress criterion

$$\boxed{Max\left(|\sigma_1|, \sigma_2|, |\sigma_3|\right) = \sigma_U}$$



- A given structural element fails when the maximum normal stress in that component reaches the material’s ultimate tensile strength.
- This criterion should only be applied to brittle materials
- It implies that the mechanism of failure is separation
- In the case of plane stress, we can draw the maximum normal stress criterion graphically. Any state of stress within the shaded area is safe

Yield Criteria for Ductile Materials



- a ductile material subjected to uniaxial tension yields and fails by slippage along oblique surfaces and is due primarily to shear stresses
- Ductile materials fail not through fracture, but through deformation.
- plastic deformation initiated at the yield strength takes place through shear deformation, it is natural to expect failure criteria to be expressed in terms of shear stress
- We therefore cast failure criteria in terms of yield:

Von Mises criterion (distortion energy criterion)

Von Mises Criterion

- This criterion for failure of ductile materials is derived from strain energy considerations and states that yielding occurs when:

$$\boxed{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]=\sigma_Y^2}$$

- To make determining the stress state for failure analysis simpler, we can calculate an equivalent Von Mises stress for each point in the structure.

$$\boxed{\sigma_M \equiv \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}}$$

To determine whether a structural component will be safe under a given load, we should calculate the stress state at all critical points of the component and particularly at all points where stress concentrations are likely to occur.

Safety factor

- We can describe how close a material in a structure is to its failure point using the safety factor.
- The safety factor compares the respective yield strength to the respective maximum or equivalent stress
- For the Von Mises safety factor we get:

$$S_M = \eta_M = \frac{\sigma_Y}{\sigma_M}$$

- sometimes the safety factor is also written as (e for equivalent):

$$\eta_e = \frac{\sigma_Y}{\sigma_e}$$